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# Computer Algebra Algorithms and Routines for the Computation of Conservation Laws and Fixing of Gauge in Differential Expressions

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In this paper we present three different approaches for the determination of conservation laws. For three corresponding REDUCE computer algebra programs CONLAW1/2/3 the necessary subroutines are described. All three programs use subroutines which remove redundant functions and constants in the general solution of the conservation law conditions. The corresponding algorithm is explained. Such a program is not only applicable in the context of computing conservation laws but whenever redundancy in differential expressions is to be removed or gauge freedom to be fixed.

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## 1. Introduction

The determination of conservation laws for single or systems of partial differential equations (PDEs) and of first integrals for ordinary differential equations (ODEs) is of interest for the exact solution of these DEs, for their understanding and classification and for supporting their numerical solution. In this paper we outline three computer algebra programs implemented in REDUCE for the computation of conservation laws. We also explain in detail a procedure that fixes the redundancy of arbitrary functions and constants in differential expressions. This procedure is used to extract individual conservation laws from the general solution of conservation-law-determining equations.

In this paper we will not make the simplifying assumption that differential equations  $\Delta = 0$  for which we want to find conservation laws result from a variational principle. In this case any variational symmetry of the Lagrangian provides a conservation law as is known from Noether's theorem. In fact, we will not make any restrictive assumptions, which leaves us to solve  $\text{Div } P = 0$  either directly or to determine the characteristic functions of the conservation laws or to do both at once. A more application-oriented

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comparison of these different approaches will be described elsewhere (Wolf, 1998). In this paper we concentrate on the computer algebra aspects.

The following section discusses the mathematical background of the three approaches. This is followed by a section on the computer algebra side of computing conservation laws. In this section the need for fixing redundancy in general solutions of conservation law conditions is motivated. A corresponding algorithm is explained in Section 4 together with other procedures that are more specific for the computation of conservation laws. In Section 5 examples are given that demonstrate in which sense the programs CONLAW1/2/3 are superior to other computer algebra programs that compute conservation laws. In two further sections the three methods are compared with each other and the syntax of the three programs is given.

## 2. The Mathematical Problem and the Three Approaches

In the following we adopt the notation in Olver (1986). Independent variables will be denoted by  $x = (x^1, x^2, \dots, x^p)$ . The differential equations for which we seek conservation laws are  $\Delta(x, u^{(n)}) = 0$  (i.e.  $\Delta_1 = 0, \dots, \Delta_q = 0$ ), for  $q$  functions  $u = (u^1, u^2, \dots, u^q)$ ,  $u^{(n)}$  denoting  $u$ -derivatives of order up to  $n$ . The conservation law that is to be fulfilled by solutions of  $\Delta = 0$  is  $\text{Div } P = \sum_{i=1}^p D_i P^i = 0$  for a current  $P = (P^1, \dots, P^p)$  and total derivatives  $D_i$ . In this paper local conservation laws are considered where components  $P^i$  are expressions in  $x^j, u^{(k)}$  and where  $\text{Div } P = 0$  has to be satisfied identically in  $x^j, u^{(k)}$  modulo  $\Delta = 0$ . We will use  $J$  as a multiple index denoting partial derivatives, for example,  $u_J^\alpha$  will stand for an arbitrary partial derivative, like  $\partial^k u^\alpha / (\partial x^1 \partial x^2 \dots)$ .

A first approach to compute conservation laws is to solve

$$\text{Div } P = 0 \quad \text{modulo } \Delta = 0 \quad (1)$$

directly. The components of the conserved current  $P^1, \dots, P^p$  that are to be calculated are allowed to depend on all independent variables  $x^i$ , the dependent variables  $u^\alpha$  and their derivatives  $u_J^\alpha$  up to some order.

We are not interested in trivial conservation laws  $P = \text{curl } V$  but rather in conservation laws that are obeyed by the solutions of  $\Delta = 0$ . Therefore we use  $\Delta = 0$  to eliminate some of the so-called jet-variables  $u_J^\alpha$  by substituting them in the determining conditions (1). This implies that condition (1) has to be fulfilled identically in fewer variables. Thus, the condition becomes less restrictive and so may have additional solutions apart from  $P = \text{curl } V$ . These extra non-trivial conservation laws are the ones of interest. We therefore assume  $\Delta = 0$  can be solved for leading derivatives  $u_J^\alpha$  so that these and all their partial derivatives that occur in (1) can be substituted.

The conserved current  $P^i$  is defined only modulo  $\Delta = 0$ , because the conservation laws are to be obeyed by solutions of  $\Delta = 0$ . We therefore assume, without loss of generality, that the  $P^i$  do not depend on the  $u$ -derivatives that we substitute. This fixes another kind of equivalence of conservation laws.

Other approaches calculate characteristic functions  $Q^\nu$ . In Olver (1986, p. 272) it is shown that, for a totally non-degenerate system  $\Delta_\nu = 0$ , each equivalence class of conservation laws  $\text{Div } P = 0$  (i.e. conserved currents differing only by a curl) is determined uniquely by characteristic functions  $Q^\nu$  satisfying

$$\text{Div } P = \sum_{\nu} Q^\nu \Delta_\nu \quad (2)$$

identically in  $all\ x^i, u^\alpha, u_J^\alpha$ . Equation (2) is not solved by simply eliminating  $Q^1$  in terms of  $P$  and  $\Delta$  and the other  $Q^\nu$  as  $Q^1$  would be singular for solutions of  $\Delta = 0$ . To avoid this problem and because the  $Q^\nu$  are only unique modulo  $\Delta = 0$ , we can ignore dependencies of  $Q^\nu$  on leading  $u$ -derivatives in  $\Delta = 0$  and any of their derivatives. One way to calculate the  $Q^\nu$  is to use the basic property of the Euler operators  $E_\nu = \sum_J (-D)_J \partial / \partial (u_J^\nu)$ , namely to give zero if and only if they act on an expression which is a divergence.

Applying the Euler operator on (2) gives

$$0 = \sum_{\mu, J} (-D)_J \left( \frac{\partial(Q^\mu \Delta_\mu)}{\partial(u_J^\nu)} \right) \quad \text{for all } \nu. \quad (3)$$

Using  $\Delta_\nu = 0$  to simplify the condition one obtains as necessary (but not sufficient) determining conditions for the  $Q^\nu$ :

$$0 = \sum_{\mu, J} (-D)_J \left( Q^\mu \frac{\partial \Delta_\mu}{\partial(u_J^\nu)} \right) \quad \text{for all } \nu. \quad (4)$$

The three methods and programs discussed in this paper aim at solving (1), (2) and (4) identically in  $x^i, u_\alpha, u_J^\alpha$ . The program **CONLAW1** tries to solve (1) for  $P^i$ , **CONLAW2** tries to solve (4) for  $Q^\nu$  and finally **CONLAW3** tries to solve (2) for  $P^i$  and  $Q^\nu$ .

Any two of the three approaches (1)–(4) differ either in the number of conditional equations to be solved, the order of these equations, the number of functions to be determined, the number of independent jet-variables, or the degree of an ansatz for  $P, Q$  that is necessary in order to obtain the same conservation laws.

To obtain solutions of (1)–(4) we assume bounds on the order of the  $u$ -derivatives on which the  $P^i$  and  $Q^\nu$  may depend. For (1) we assume a bound for  $P^i$  and for (2),(4) we assume a bound for  $Q^\nu$ . Bounds for the remaining unknown functions follow. Differentiations done in all three conditions (1)–(4) introduce jet-variables ( $u$ -derivatives) on which none of the unknown functions depend.

As the conditions have to be satisfied identically in these variables as well, over-determined PDE systems result. Those are investigated with the computer algebra package **CRACK** (Wolf and Brand, 1992; Wolf, 1996).

### 3. The Computer Algebra Problem

The main computer algebra problem is to solve the over-determined conditions (1)–(4). Steps for solving this problem include separation, integration, application of integrability conditions (differential Gröbner basis), solution of ODEs and other steps, all of which are described in Wolf and Brand (1992). The success rate of **CRACK** in solving over-determined systems is high if the system is linear (like (1)–(4) which are linear in  $P^i, Q^\nu$ ) and not too big. Examples of what is currently possible are given in Section 5. Any unsolved equations are returned. For example, when the conservation laws of the Burgers equation are investigated in Section 5, the heat equation remains as an unsolved condition and is returned. Complexity issues are discussed in Wolf (1998).

From the general solution of (1),(2) or (4) a single conservation law is extracted by collecting all terms involving one of the arbitrary constants or arbitrary functions in the solution. Any redundant constants would give conservation laws which are not independent of each other.

Redundant constants and functions may result because in the process of solving the over-determined system, there is no general rule for deciding whether integrations or substitutions should have a higher priority. There are examples requiring a higher priority for each of them. It therefore may happen that two equations are integrated which are not independent of each other and, therefore, the constants or functions of integration are not independent of each other. As a consequence, the final general solution could have redundant arbitrary constants and functions. For example, in the expression  $c_1(x)t + c_2xt + c_3$ , with independent variables  $x, t$  and arbitrary function  $c_1(x)$  and arbitrary constants  $c_2, c_3$ , the constant  $c_2$  is redundant as it can be absorbed by  $c_1(x)$  through  $c_1(x) \rightarrow c_1(x) - c_2x$ . In this paper we will also refer to redundancies as gauge freedom.

Another application of redundancy recognition is the solution of PDE systems with some gauge freedom where the problem is to eliminate any gauge freedom from the general solution of this system. This can be accomplished by including in the solution terms representing the complete gauge freedom. For example, in the case of conditions (1)

- the general solution could be augmented by  $\text{curl } V$  and  $V$  be added to the list of free constants and functions;
- redundant constants and functions could be dropped, i.e. constants and functions that could be absorbed by  $V$ ;
- finally  $V$  could be dropped from the list of free functions.

In this way trivial conservation laws could be filtered out as the free constants and functions corresponding to such conservation laws would have been absorbed by  $V$ .

In the case of computing conservation laws, one could easily drop trivial conservation laws after they have been computed by checking  $\text{Div } P = 0$  identically in all jet-variables. Such a simple test to eliminate gauge freedom might not be available for other problems. Therefore a general algorithm is described below.

#### 4. Subroutines

In the following subsections we describe an algorithm for fixing redundancy in the general solution of a PDE problem (and by that extracting conservation laws from the general solution of conditions (1)-(4)), subroutines that compute  $Q^\nu$  from  $P^i$  and  $P^i$  from  $Q^\nu$ , and subroutines that simplify  $P^i$ .

##### 4.1. IDENTIFYING REDUNDANT CONSTANTS AND FUNCTIONS IN SOLUTIONS

This and the following section, 4.2, are self-contained and could be read independently of the rest of the paper. In them we consider solutions of arbitrary differential equations, having in mind the conditions (1)-(4) (and less the equations  $\Delta = 0$ , which we cannot solve and which therefore are investigated for conservation laws). To avoid confusion we call these equations  $\Omega(f_a, x^i) = 0$  and these are to be solved for the functions  $f_a(x^i)$ . In the case of computing conservation laws, the equations  $\Omega = 0$  are the equations (1)-(4), the  $f_a$  are the  $P^i$  and/or  $Q^\mu$ , and the  $x^i$  now include all jet-variables. This notation is only used in Sections 4.1 and 4.2.

In this subsection we wish to discuss the problem of fixing gauge freedom, if only a PDE-solver that could produce solutions with possible redundancies is available. Before giving the steps in a summary, the algorithm is explained using simple examples.

Given a system of DEs  $\Omega(f_a, x^i) = 0$  that is to be solved for the functions  $f_a(x^i)$ , we assume

$$f_b = F_b(x^i, g_c) \quad (5)$$

to be a general solution where the  $F_b$  are differential expressions in  $x^i, g_c$  where the  $g_c$  are arbitrary constants and functions of  $x^i$  (which, for example, resulted through integration).

The question is how to specify as many as possible of the  $g_c$  to fix any redundancy among them but not to lose the generality of the solution. The idea behind the algorithm is the following.

If two sets of constants and functions  $g_c$  and  $\bar{g}_c$  in solution (5) differ only by a gauge then for both sets solution (5) would still be identical in the  $x^i$ . Therefore, the determining conditions for the redundancy will be the invariance condition for the general solution, i.e.

$$F_b(x^i, g_c) - F_b(x^i, \bar{g}_c) = 0 \quad (6)$$

identically in the  $x^i$  where the  $\bar{g}_c$  are new functions, each  $\bar{g}_a$  having the same variable dependence as the  $g_a$  with the same lower index.

EXAMPLE. We assume the general solution of some problem for a single function  $f(x^1, x^2, x^3)$  being computed is

$$f = x^1 g_1(x^2) + g_2(x^1, x^2) + x^3 g_3 \quad (7)$$

with the arbitrary functions  $g_1(x^2), g_2(x^1, x^2)$  and the arbitrary constant  $g_3$ .<sup>†</sup>

For the solution (7), the invariance condition (6) is

$$x^1 g_1(x^2) + g_2(x^1, x^2) + x^3 g_3 - x^1 \bar{g}_1(x^2) - \bar{g}_2(x^1, x^2) - x^3 \bar{g}_3 = 0. \quad (8)$$

Seen as a condition for the functions  $g_1(x^2), \bar{g}_1(x^2), g_2(x^1, x^2), \bar{g}_2(x^1, x^2)$  and the constants  $g_3, \bar{g}_3$  it is solved by separation of variables (here  $x^3$ ):

$$\begin{aligned} 0 &= g_3 - \bar{g}_3 \\ 0 &= x^1 g_1(x^2) + g_2(x^1, x^2) - x^1 \bar{g}_1(x^2) - \bar{g}_2(x^1, x^2), \end{aligned}$$

which has as a general solution

$$\begin{aligned} \bar{g}_3 &= g_3 \\ \bar{g}_2(x^1, x^2) &= g_2(x^1, x^2) + x g_1(x^2) - x \bar{g}_1(x^2), \end{aligned} \quad (9)$$

$\bar{g}_1(x^2), g_1(x^2), g_2(x^1, x^2), g_3$  are free. If it is possible to compute the general solution of the invariance condition (6) then all  $g_c$  for which  $g_c$  itself or  $\bar{g}_c$  are computed (occur on the left-hand side of this general solution) are *essential* free constants and functions of the general solution (5). In solution (7) the  $g_2(x^1, x^2), g_3$  are essential.

<sup>†</sup>The general solution to conservation law problems usually involves only constants, like in Examples 1 and 3 in Section 5. Example 2 in Section 5 only involves functions because this is a rare case of a linearizable equation—the Burgers equation.

The remaining  $g_c$  which occur together with their counterpart  $\bar{g}_c$  on right-hand sides of the general solution of the invariance condition are *redundant*. They can be chosen freely (like  $g_1(x^2)$  in solution (7) of the example). Any change in them can be compensated by a change in the essential functions; (9) shows how  $\bar{g}_2(x^1, x^2)$  has to be adapted if  $g_1(x^2), \bar{g}_1(x^2)$  are to be changed.

Although the solvability of (6) for  $g_a, \bar{g}_b$  cannot be guaranteed, this should in practice not be a problem for the following reasons.

- Under normal circumstances when (5) is a solution to some PDE(s), functions  $g_c$  appearing on the right-hand side have to be functions of fewer variables than  $f_b$  on the left-hand side. The invariance condition (6) which is to be satisfied in *all* independent variables  $x^i$  identically is, therefore, a highly over-determined problem for the functions  $g_c, \bar{g}_c$  of only a subset of variables of (6). Even if the problem to which (5) is the solution is under-determined, and so consequently some  $g_c$  have the same variable dependence as the  $f_b$ , still their number must be smaller than the number of  $f_b$ . In any case, the invariance condition (6) is an over-determined set of conditions for the  $g_c, \bar{g}_c$  and therefore well suited to be solved by the package CRACK.
- Although it is over-determined, the invariance condition (6) has always at least one trivial solution,  $\bar{g}_c = g_c$ , for all  $c$ . This is the general solution of the invariance condition if no redundancy exists.
- If the equations  $\Omega = 0$  are linear in  $f_a$ , then their solution is linear in the arbitrary functions  $g_c$ , i.e. equation (6) is linear in the  $g_c, \bar{g}_c$ . This is the case for the computation of conservation laws.<sup>†</sup>
- If the equations  $\Omega = 0$  are nonlinear in  $f_a$ , then solving (6) should still be easier than solving  $\Omega = 0$ , an equation which we assumed was possible to solve.

In the following example we show how new redundancy can result from solving the invariance condition. The example also illustrates that it may be possible to compute gauge freedom but that it is impossible to remove it.

EXAMPLE. As mentioned above, the more under-determined a problem is, the more difficult it is in general to fix redundancy in its general solution. In this respect the following well-known simple problem (one condition for three unknowns) is comparatively hard. The following equation

$$0 = \text{Div} f = \partial_1 f_1 + \partial_2 f_2 + \partial_3 f_3 \quad (10)$$

for the functions  $f_1, f_2, f_3$  of the variables  $x^1, x^2, x^3$  has the general solution

$$\begin{aligned} f_1 &= \partial_3 g_2 - \partial_2 g_3 \\ f_2 &= \partial_1 g_3 - \partial_3 g_1 \\ f_3 &= \partial_2 g_1 - \partial_1 g_2 \end{aligned} \quad (11)$$

where  $g_1, g_2, g_3$  are arbitrary functions of  $x^1, x^2, x^3$ . (Equation (10) is *not* a conservation law problem. It is not to be solved modulo some equation, but describes a static source free field.)

<sup>†</sup>Conditions become nonlinear if  $\Delta = 0$  does contain parameters and we want to calculate their value such that conservation laws exist.

The general solution (11) must have gauge freedom as it has the same number of free functions  $g_i$  of all variables as there are functions  $f_i$  to be determined initially. To solve the invariance conditions

$$0 = \partial_3 g_2 - \partial_2 g_3 - \partial_3 \bar{g}_2 + \partial_2 \bar{g}_3 \quad (12)$$

$$0 = \partial_1 g_3 - \partial_3 g_1 - \partial_1 \bar{g}_3 + \partial_3 \bar{g}_1 \quad (13)$$

$$0 = \partial_2 g_1 - \partial_1 g_2 - \partial_2 \bar{g}_1 + \partial_1 \bar{g}_2 \quad (14)$$

a computer algebra program could do the following steps:

- Solve (12) by introducing a new function  $k_1(x^1, x^2, x^3)$  with

$$\bar{g}_2 - g_2 = \partial_2 k_1, \quad \bar{g}_3 - g_3 = \partial_3 k_1. \quad (15)$$

- Solve (13) by substituting  $g_3$  from (15) and integrating it with respect to  $x^3$  with a new function  $k_2(x^1, x^2)$  to obtain

$$\bar{g}_1 - g_1 = \partial_1 k_1 + k_2.$$

- Equation (14) finally requires  $\partial_2 k_2 = 0$ , giving the general solution

$$\bar{g}_1 = g_1 + \partial_1 k_1 + k_2 \quad (16)$$

$$\bar{g}_2 = g_2 + \partial_2 k_1 \quad (17)$$

$$\bar{g}_3 = g_3 + \partial_3 k_1 \quad (18)$$

with  $k_1 = k_1(x^1, x^2, x^3)$ ,  $k_2 = k_2(x^1)$ .

This example shows how solving the invariance condition may require solving differential equations; therefore, the solution may involve extra redundant functions, here  $k_2$ . The free function  $k_1$  is a function of all variables. Therefore, if  $k_1$  occurs in one of equations (16)–(18) only algebraically then, at least locally, this equation could be solved for  $k_1$ , and  $k_1$  could be substituted so that only two of the  $g_c$  are essential.

The situation in solution (16)–(18) is different.  $k_1$  cannot be solved for and the redundancy cannot be fixed. This deficiency is not only a technical aspect. If a derivative of  $k_1$  could be renamed to  $\tilde{k}_1$  in order to have  $\tilde{k}_1$  appearing algebraically in at least one equation and everywhere else only  $\tilde{k}_1$  (not  $k_1$ ) or derivatives of  $\tilde{k}_1$  then this would be done in the corresponding routine in CRACK. However, in (16)–(18)  $k_1$  comes with different derivatives; therefore, it is not a deficiency of the algorithm or computer program that gauge freedom cannot be fixed explicitly in a general solution like (11).

The following summary is a list of steps as they are performed in the program CRACK to fix gauge freedom in a given solution (5)

$$f_b = F_b(x^i, g_c)$$

to some differential problem:

- Formulate a set of conditions (6)

$$F_b(x^i, g_c) - F_b(x^i, \bar{g}_c) = 0$$

where  $\bar{g}_c$  are new functions, each  $\bar{g}_c$  having the same variable dependence as  $g_c$ . Regard equation (6) as a system of equations for the set of unknown functions  $\{g_c, \bar{g}_c\}$ , to be satisfied identically in the  $x^i$ .

- Find the general solution of the system (6) as

$$\tilde{g}_c = G_c(x^i, h_d) \quad (19)$$

where  $\tilde{g}_c$  is a subset of  $\{g_a, \bar{g}_b\}$ , and  $G_c$  are algebraic or differential expressions in the functions  $h_d$  which are the remaining  $\{g_a, \bar{g}_b\}$  and extra constants and functions of integration. (In the example above the  $G_c$  are the right-hand sides of (16)–(18), the  $\tilde{g}_c$  are  $\bar{g}_1, \bar{g}_2, \bar{g}_3$  and the  $h_d$  are  $g_1, g_2, g_3, k_1, k_2$ .) The  $h_d$  are arbitrary. Any function  $g_a$  or  $\bar{g}_a$  appears only once on a left-hand side of (19) or only on the right-hand sides.

- If for any index  $c$ , both  $g_c$  and  $\bar{g}_c$  appear only on the right-hand sides of (19) then  $g_c$  is redundant and can be set to zero in all  $F_b$  in (5) and all  $G_c$  in (19).
- The system (19) will now be solved for the  $\bar{g}_c$ : if for any index  $c$ , both  $g_c$  and  $\bar{g}_c$  appear only on the left-hand sides of (19) in the equations  $g_c = G_c$  and  $\bar{g}_c = \bar{G}_c$  then these two equations are replaced by  $\bar{g}_c = g_c - G_c + \bar{G}_c$  in (19).
- If for any index  $c$ ,  $g_c$  appears on a left-hand side of (19) and  $\bar{g}_c$  appears only on the right-hand sides then the equation with left-hand side  $g_c$  is solved for  $\bar{g}_c$  and replaced by the new equation  $\bar{g}_c = \bar{G}_c(g_c, \dots)$ . With this new equation  $\bar{g}_c$  is substituted on any right-hand side of (19).
- There remains only the case of  $\bar{g}_c$  being on the left-hand side of an equation and  $g_c$  being on the right-hand sides. The system (19) now has the form

$$\bar{g}_c = \bar{G}_c(x^i, g_a, \bar{h}_b) \quad (20)$$

where  $\bar{h}_b$  are arbitrary constants and functions of integration which arose during the solution of (6). The  $\bar{g}_c$  do not occur on the right-hand sides as they would be redundant and would have been set to zero otherwise.

- Free constants and functions  $\bar{h}_b$  that occur only algebraically on the right-hand side of at least one equation and that depend on all independent variables in that equation will be chosen to make as many  $\bar{G}_c$  as possible zero. Also, in such a case the redundant  $g_c$  is set to zero in (5) and (20). (In equation (16)  $k_2$  could not be used as  $k_2$  does not depend on all the variables and  $k_1$  could not be used as it does not occur algebraically in at least one equation. Also, a substitution of a derivative of  $k_1$  by, say  $k_1$ , would introduce integrals and is therefore not possible.) As we do not have to know  $\bar{h}_b$  explicitly, it is enough to find equations in (20) which include such a  $\bar{h}_b$ . Assuming local solvability of  $\bar{G}_c = 0$  for  $\bar{h}_b$  we conclude the redundancy of  $g_c$ .
- Finally, all remaining  $\bar{h}_b$  which cannot be used to make a right-hand side zero are set to zero themselves (in example (16)–(17) these are  $k_1, k_2$ ). The final form of (20)

$$\bar{g}_c = \bar{G}_c(x^i, g_a) \quad (21)$$

provides substitutions which turn the old solution (5) ( $f_b = F_b(x^i, \bar{g}_c)$ ) after the  $g_c$  have been renamed  $\bar{g}_c$ ) into a new solution  $f_b = \tilde{F}_b(x^i, g_c)$  for which as much redundancy as possible is fixed.

#### 4.2. IDENTIFYING REDUNDANT CONSTANTS AND FUNCTIONS IN SOLUTIONS AND REMAINING UNSOLVED EQUATIONS

The above steps for fixing redundancy are not only applicable once a general solution of a PDE(system)  $\Omega(f_a, x^i) = 0$  has already been found. If a preliminary solution is



known but some equations remain to be solved, then fixing at least some redundancy is possible. For example, the computation of conservation laws for the Burgers equation in Section 5 returns a solution together with the heat equation which remains unsolved.

What could the fixing of redundancies mean under these circumstances? Substitutions that find redundancies in  $g_c$  in a preliminary solution  $f_b = F_b(x^i, g_c)$  and in remaining differential equations  $D(x^i, g_c) = 0$  which have to be satisfied by  $g_c$  should leave the solution completely unchanged but should be allowed to change the remaining equations  $D = 0$  into a set of equivalent equations. This is, in general, a much harder problem than we had before as it involves equivalence of PDE systems. A simpler question is to find redundancy fixing substitutions that leave the solution *and* equations unchanged. This is done by extending redundancy conditions (6) by  $D(x^i, g_c) - D(x^i, \bar{g}_c) = 0$ .

Although this simplified version of fixing redundancies seems to miss potential redundancies, we have not encountered such a case yet, given that our remaining equations are at most polynomially nonlinear (for linear conservation law calculations), unknown functions are not nested, or depending on expressions (like  $f(x+y)$ ), and in each equation there is at least one function of all variables.

EXAMPLE. Using the above steps to delete redundancies in the preliminary solution

$$f = g_1(x^1, x^3) - g_2(x^1, x^2) - g_3(x^2), \quad (22)$$

the remaining condition

$$g_2(x^1, x^2) + g_3(x^2) + g_4(x^3) = 0 \quad (23)$$

would show that  $g_3$  is redundant but  $g_4$  not. The reason for not recognizing that, after substituting  $g_2$  from (23) into (22),  $g_4$  can be absorbed by  $g_1$  is that equation (23) does not involve a function of all the variables  $x^1, x^2, x^3$ .

A situation like this cannot occur in practice, as the program **CRACK** has routines to handle equations in which there is no function of all variables. An equation like (23) can therefore not remain unsolved. **CRACK** would find  $g_4 = \text{constant}$  (and  $g_2 + g_3 = \text{constant}$ ) and then  $g_4$  would be absorbed.

To look at the problem in terms of redundancy fixing substitutions that transform (23) into an equivalent condition, a computer program would have to be able to see that equation (23) is equivalent to

$$g_2(x^1, x^2) + g_3(x^2) + g_5 = 0, \quad \text{and} \quad g_4(x^3) = g_5 = \text{constant}$$

and then  $g_5$  would be found to be redundant.

The possibility of fixing gauge freedom even in the presence of yet unsolved equations opens the possibility of running a gauge-fixing step during the process of solving overdetermined PDE systems (which is currently not done). As a consequence the number of unknown functions could be reduced and the remaining equations simplified.

#### 4.3. COMPUTING CHARACTERISTIC FUNCTIONS FROM CONSERVED CURRENTS

The remainder of the paper is concerned with the computation of conservation laws. In this subsection we explain how to compute the characteristic functions  $Q$  from the conserved current  $P$ .

Approach (1) to computing conservation laws is attractive compared with (2),(4) as it generates only one PDE to be solved which is of first order and involves fewer jet-variables than approach (2) because it is computed modulo  $\Delta = 0$ . Also, it has fewer functions to compute than approach (2). A negative aspect is that it provides only the conserved current  $P$  and not the characteristic functions  $Q$ .

If expressions  $Q^{\nu J}$  in relation (24) below are known then partial integrations (25) yield the characteristic functions  $Q^\nu$  and the corresponding conserved current  $P - R$ :

$$\begin{aligned} \text{Div } P &= 0 \text{ modulo } \Delta_\nu = 0 \leftrightarrow \\ \exists Q^{\nu J} : \text{Div } P &= \sum_{\nu, J} Q^{\nu J} \Delta_\nu^{(J)} \quad (\text{identically in all } x, u_J^\alpha) \end{aligned} \quad (24)$$

$$\begin{aligned} &= \sum_{\nu, J} D_J(Q^{\nu J} \Delta_\nu) - D_J(Q^{\nu J}) \Delta_\nu \quad (\text{repeatedly}) \\ &= \text{Div } R + \sum_\nu Q^\nu \Delta_\nu. \end{aligned} \quad (25)$$

Equation (24) cannot be regarded as a linear algebraic equation to determine  $Q^{\nu J}$  as there is the additional requirement that the  $Q^{\nu J}$  are non-singular for solutions of  $\Delta = 0$ . Instead,  $\text{Div } P$  is calculated and substitutions of a different form from before are made. For example, if conservation laws for the Harry–Dym equation,  $0 = \Delta = u_t - u^3 u_{xxx}$ , are investigated and if for the derivation of (1) the prior substitutions  $u_t = u^3 u_{xxx}$ ,  $u_{tx} = (u^3 u_{xxx})_x, \dots$  had been made then now the substitutions would be  $u_t = \Delta + u^3 u_{xxx}$ ,  $u_{tx} = \Delta_x + (u^3 u_{xxx})_x, \dots$  which provide the right-hand side of (24). The computation of  $Q^\nu$  and  $P^i - R^i$  from  $P^i$  is part of CONLAW1.

#### 4.4. COMPUTING CONSERVED CURRENTS FROM CHARACTERISTIC FUNCTIONS

The inverse computation is necessary in CONLAW2 where the conserved current  $P^i$  has to be computed from  $Q^\mu$  by integrating  $\text{Div } P = \sum_\nu Q^\nu \Delta_\nu$ .

A direct way is based on a formula given by Anco and Bluman (1997):

$$P^i = \int_0^1 \frac{d\lambda}{\lambda} (S^i(u) + N_\mu^i(u) u^\mu + N_\mu^{ij}(u) D_j u^\mu + \dots) |_{u \rightarrow \lambda u} \quad (26)$$

$$S^i(u) = Q^\nu \frac{\partial \Delta_\nu}{\partial u_i^\mu} u^\mu + Q^\nu \frac{\partial \Delta_\nu}{\partial u_{ij}^\mu} u_j^\mu - \left( Q^\nu \frac{\partial \Delta_\nu}{\partial u_{ij}^\mu} \right)_j u^\mu + \dots \quad (27)$$

$$N_\mu^i(u) = \frac{\partial Q^\nu}{\partial u_i^\mu} \Delta_\nu - \left( \frac{\partial Q^\nu}{\partial u_{ij}^\mu} \Delta_\nu \right)_j + \left( \frac{\partial Q^\nu}{\partial u_{ijk}^\mu} \Delta_\nu \right)_{jk} - \dots \quad (28)$$

$$N_\mu^{ij}(u) = \frac{\partial Q^\nu}{\partial u_{ij}^\mu} \Delta_\nu - \left( \frac{\partial Q^\nu}{\partial u_{ijk}^\mu} \Delta_\nu \right)_k + \left( \frac{\partial Q^\nu}{\partial u_{ijkl}^\mu} \Delta_\nu \right)_{kl} - \dots \quad (29)$$

where summation is performed over double indices.

These formulas are programmed in the more compact form

$$\begin{aligned} V &= Q^\nu \Delta_\nu, \\ W^i &= n(i) u^\mu \frac{\partial V}{\partial u_i^\mu} + \end{aligned}$$

$$\begin{aligned}
& n(ij)(u_j^\mu - u^\mu D_j) \frac{\partial V}{\partial u_{ij}^\mu} + \\
& n(ijk)(u_{jk}^\mu - u_j^\mu D_k + u^\mu D_j D_k) \frac{\partial V}{\partial u_{ijk}^\mu} + \\
& \vdots \\
T^i &= x^i \int_0^1 d\lambda \lambda^{p-1} V|_{u \rightarrow 0, x \rightarrow \lambda x} \\
P^i &= T^i + \int_0^1 \frac{d\lambda}{\lambda} W^i|_{u \rightarrow \lambda u}
\end{aligned} \tag{30}$$

where in  $W^i$  it is summed over equal indices (not the  $i, j, k, \dots$  in  $n(i, j, \dots)$ ) with  $i \leq j \leq k \leq \dots$  and  $n(i, j, \dots) = \prod_a r_a! / (\sum_b r_b)!$  with  $r_a$  being the multiplicities of different arguments  $i, j, \dots$  of  $n$  (e.g.  $n(i) = 1, n(i, i) = 1, n(1, 2) = 1/2$ ) which also occur in (27)–(29).  $p$  is the number of variables  $x^i$  and  $T^i$  are non-zero only if  $u \equiv 0$  does not solve  $\Delta = 0$  ( $T^i$  have to enter (26) in that case as well).

Despite being an elegant formula there may be problems in computing the integral analytically. More seriously, the integral may be singular for  $\lambda = 0, 1$ . That is the case, for example, for the non-polynomial characteristic functions of the Harry–Dym equations in the next section.

If taking the limit  $u \rightarrow 0$  in  $V|_{u \rightarrow 0, x \rightarrow \lambda x}$  or taking the limit  $u \rightarrow \lambda u$  in  $W^i|_{u \rightarrow \lambda u}$  produces a singularity then this can be circumvented by choosing a  $\tilde{u}(x)$  (e.g.  $\tilde{u} = 1$  or  $\tilde{u} = x$ ) such that after replacing  $u \rightarrow \lambda u + (1 - \lambda)\tilde{u}$  the limits  $V|_{u \rightarrow \tilde{u}, x \rightarrow \lambda x}$  and  $W^i|_{u \rightarrow \lambda u + (1 - \lambda)\tilde{u}}$  become non-singular (Anco and Bluman, 1998).

Because of these potential difficulties, the default procedure to compute  $P^i$  is to use the integration module of **CRACK** to  $x^1$ -integrate  $\sum_\nu Q^\nu \Delta_\nu$ , to  $x^2$ -integrate the remaining unintegrated terms, and so on. In case terms remain after the last  $x^p$ -integration, the process is restarted on the remaining terms until all terms are integrated, or at most a fixed number of times. If this method does not work because not all determining conditions can be solved as, for example, for the Burgers equation below then (30) is used.

If both methods fail then the program concludes that solutions  $Q$  of (4) represent adjoint symmetries and not characteristic functions of conservation laws.

#### 4.5. THE SIMPLIFICATION OF $P$ IN TWO VARIABLES

After deleting trivial conservation laws and identifying equivalent conservation laws through the computation of characteristic functions  $Q$  it remains to simplify the conserved current  $P$  through the addition of some curl:  $P \rightarrow P + \text{curl } V$ . This is done if there are only two independent variables, say  $x^1, x^2$ . The aim is to lower the order of  $x^2$ -derivatives in  $P^1$  through changes  $P^1 \rightarrow P^1 - D_2 R$ ,  $P^2 \rightarrow P^2 + D_1 R$ .  $R$  is found by repeated partial integration of terms in  $P^1$  with highest  $x^2$ -derivatives of  $u$ . For that, partial integration routines of **CRACK** are used which are limited in applicability to expressions which are at most polynomially nonlinear in  $u$  and derivatives of  $u$ .

## 5. Examples

Computation times refer to a 24 MB REDUCE 3.6 session under LINUX on a 133 MHz Pentium PC with the January 1998 version of CRACK.

EXAMPLE 1. The advantage of using the package CRACK for solving determining equations is that they can be PDEs and do not have to be restricted to algebraic equations for coefficients of a polynomial ansatz for the conservation law. As a consequence, it is possible to find non-polynomial conservation laws and conservation laws that have an explicit  $x^i$  dependence. An example is the Harry–Dym equation

$$\Delta = u_t - u^3 u_{xxx}, \quad u = u(t, x)$$

which was used below to substitute  $u_t$  and derivatives of  $u_t$ . These calculations were done with CONLAW1. In the following, the calculation to find conservation laws with  $P^t$  of order 0 is described. The results for this case and the higher-order cases are given afterwards.

If  $P^t$  is of order 0, i.e.  $P^t = P^t(x, t, u)$ , condition (1),  $\text{Div } P = 0$ , will involve  $u_t$  from  $D_t P^t$ . After substitution of  $u_t$  from the Harry–Dym equation it will have  $u_{xxx}$  as the highest  $u$ -derivative which has to match the highest  $u$ -derivative in  $D_x P^x$ . Therefore, CONLAW1 makes the ansatz  $P^t = P^t(x, t, u)$ ,  $P^x = P^x(x, t, u, u_x, u_{xx})$  for conservation laws with  $P^t$  of order 0, then formulates the condition  $\text{Div } P = 0$  as

$$u_{xxx}(u^3 P_u^t + P_{u_{xx}}^x) + P_t^t + P_x^x + P_u^x u_x + P_{u_x}^x u_{xx} = 0 \quad (31)$$

and uses CRACK to solve it. As  $P^t, P^x$  do not depend on  $u_{xxx}$ , a separation performed by CRACK returns

$$0 = u^3 P_u^t + P_{u_{xx}}^x \quad (32)$$

$$0 = P_t^t + P_x^x + P_u^x u_x + P_{u_x}^x u_{xx}. \quad (33)$$

$P^t$  does not depend on  $u_{xx}$ , so integration of (32) with respect to  $u_{xx}$  introduces a function of integration  $g_1(x, t, u, u_x)$  and substitution of  $P^x$  into (33) gives an equation that can be separated with respect to  $u_{xx}$ . Two equations for  $P^t(x, t, u)$  and  $g_1(x, t, u, u_x)$  result. One of the equations can be integrated with respect to  $u_x$  with a function of integration  $g_2(x, t, u)$  such that  $g_1$  can be substituted into the other equation which then becomes separable with respect to  $u_x$ , giving four equations. One of the four equations  $3P_{xu}^t + uP_{xuu}^t = 0$  can be integrated three times to get  $u^2 P^t + g_3(t, u) + g_4(x, t) + u^2 g_5(x, t)/2 = 0$ . Substituting  $P^t$  into the other three equations yields, among two other equations,  $g_{3,uuu} = 0$  which integrated and  $g_3$  being substituted provides two equations. One of these equations can be  $u$ -integrated such that  $g_2$  can be substituted into the other equation, which in turn becomes  $u$ -separable, giving four equations.

The computation so far needed 14 steps in CRACK. Continuing this process of integrations, substitutions, separations for another 14 steps returns the solution

$$P^t = -(g_{17,x} u^2 + 2u g_{10} + 2x g_{14} + x^2 g_{15} + 2g_{16} - g_7 u^2)/(2u^2) \quad (34)$$

$$P^x = (g_{17,t} - x g_{7,t} - 2u u_{xx} g_{10} + u_x^2 g_{10} + 4u_x g_{14} - 4x u_{xx} g_{14} - 4u g_{15} + 4x u_x g_{15} - 2x^2 u_{xx} g_{15} - 4u_{xx} g_{16})/2 \quad (35)$$

with free  $g_{17}(x, t), g_7(t), g_{10}, g_{14}, g_{15}, g_{16}$ . The algorithm of Section 4.1 finds  $g_7$  to be redundant and sets it to zero which concludes the run of CRACK. In CONLAW1, the terms in the general solution returned by CRACK which involve  $g_7$  are found to fulfil  $\text{Div } P = 0$  identically. Therefore, this conservation law is trivial and is dropped. The coefficients

of the remaining four constants give four non-trivial conservation laws. As described in Section 4.3 the characteristic functions  $Q$  are computed which finally gives:

$P^t$  of order 0: time to formulate (1): 0.32s, to solve (1): 1.34 s, 28 steps:

$$\begin{aligned} 2u^{-2} \cdot \Delta &= D_t(-2u^{-1}) + D_x(u_x^2 - 2uu_{xx}) \\ 2u^{-3} \cdot \Delta &= D_t(-u^{-2}) + D_x(-2u_{xx}) \\ 2xu^{-3} \cdot \Delta &= D_t(-xu^{-2}) + D_x(2u_x - 2xu_{xx}) \\ 2x^2u^{-3} \cdot \Delta &= D_t(-x^2u^{-2}) + D_x(4xu_x - 2x^2u_{xx} - 4u) \end{aligned}$$

Similar computations for higher orders give:

$P^t$  of order 1: time to formulate (1): 0.32 s, to solve (1): 2.6 s, 49 steps:

$$(2uu_{xx} - u_x^2)u^{-2} \cdot \Delta = D_t(-u_x^2u^{-1}) + D_x((2u_tu_x - u_{xx}^2u^3 + u_{xx}u_x^2u^2 - u_x^4u/4)u^{-1})$$

$P^t$  of order 2: time to formulate (1): 1.7 s, to solve (1): 110 s, 193 steps<sup>†</sup>:

$$\begin{aligned} &(-8u_{xxx}u^3 - 16u_{xxx}u_xu^2 - 12u_{xx}^2u^2 + 12u_{xx}u_x^2u - 3u_x^4)u^{-2} \cdot \Delta \\ &= D_t((-4u_{xx}^2u^2 - 3u_{xx}u_x^5tu - u_x^4)u^{-1}) \\ &+ D_x((8u_{tx}u_{xx}u^2 + 3u_{tx}u_x^5tu - 8u_tu_{xxx}u^2 - 8u_tu_{xx}u_xu + 4u_tu_x^3 \\ &+ 4u_{xxx}^2u^5 + 4u_{xx}^3u^4 - 6u_{xx}^2u_x^2u^3 + 3u_{xx}u_x^4u^2)u^{-1}) \end{aligned}$$

Computations for higher order conservation laws of equations  $\Delta = 0$  that are of higher order, in more variables  $x^i$  and more functions  $u^\alpha$  are similar to the one described above. However, the application of integrability conditions may be necessary (as in a computation for a differential Gröbner basis) and a more general separation method may be necessary if no function depends on all variables and each variable occurs in at least one function. More difficult computations may involve several thousand steps.

EXAMPLE 2. The Burgers equation in the form

$$\Delta = u_t - u_{xx} - \frac{1}{2}u_x^2 = 0, \quad u = u(t, x) \quad (36)$$

is an example for the case that the determining equations cannot be solved completely. It has zeroth-order conservation laws

$$fe^{u/2}\Delta = D_t(2fe^{u/2}) + D_x(e^{u/2}(2f_x - fu_x)) \quad (37)$$

with  $f = f(t, x)$  satisfying the linear reverse heat equation  $f_t + f_{xx} = 0$ . The above law is also an example that CONLAW allows the computation of conservation laws with non-rational terms which is not possible with approaches based on a polynomial ansatz. A remaining linear PDE and the occurrence of free functions in the conservation law indicates linearizability of  $\Delta = 0$ , which is the case with the Burgers equation.

EXAMPLE 3. The MVDNLS equations (modified vector derivative nonlinear Schrödinger equations) describe the oblique propagation of magneto-hydrodynamic waves in warm

<sup>†</sup>The times and the number of steps depend strongly on the relative priorities of the individual steps done (integrations, substitutions, etc) which is especially the case for longer computations and therefore these measures characterize the complexity only roughly.

plasmas (Wilcox *et al.*, 1995). For functions  $u = u(t, x)$ ,  $v = v(t, x)$  and  $b = \text{constant}$  they are:

$$\Delta_1 = u_t + [u(u^2 + v^2) + bu - v_x]_x \quad (38)$$

$$\Delta_2 = v_t + [v(u^2 + v^2) + u_x]_x. \quad (39)$$

Both equations have the form of conservation laws. Using the abbreviations (introduced by hand afterwards)

$$\begin{aligned} E &= -v_x + u(u^2 + v^2) \\ F &= u_x + v(u^2 + v^2 - b) \\ G &= 2u_{xx} + 6v_x(u^2 + v^2) - 3u(u^2 + v^2)^2 - 2bu^3 \\ H &= 2v_{xx} - 6u_x(u^2 + v^2) - 3v(u^2 + v^2)^2 + 2bv^3 \\ I &= b(u^4 - v^4) + (u^2 + v^2)^3 - 2u_x^2 - 2v_x^2 \end{aligned}$$

and using equations (38), (39) to substitute for  $u_t, v_t$ , further conservation laws calculated by CONLAW2/3 have the characteristics  $\{Q^1, Q^2\}$ :

$$\begin{aligned} \{u, v\}, \quad \{E, F\}, \quad \{G, H\}, \quad (40) \\ \{(bt - 2x)E - 2tG + b(bt - x)u + v, \quad (bt - 2x)F - 2tH + b(bt - x)v - u\}, \quad (41) \end{aligned}$$

$$\{-H_x + 2uvH + (b + 2u^2)G + uI, \quad G_x + 2uvG + 2v^2H + vI\}. \quad (42)$$

CONLAW2 can compute one more conservation law with  $Q^1, Q^2$  of fourth order and 36 terms each. Run times are listed in Table 1.<sup>†</sup>

Apart from (41) these conservation laws are given in Wilcox *et al.* (1995) where a bi-Hamiltonian structure is also provided. Although an infinite sequence of conserved densities can be calculated from the resulting recursion operator, the conservation law (41) is not contained in that sequence and is new—it has an explicit  $t, x$ -dependence.

In the scope of CONLAW1 to find conservation laws with  $P^1$  of order 1 are conservation laws (40), (41).

## 6. Comparison of the Three Methods

The determining equations (1)–(4) differ in the number of functions, number of variables and their order. For example, for the MVDNLS equations (38), (39) condition (1) for conservation laws with  $P^1$  of order 2 and conditions (2), (4) for conservation laws with  $Q^\mu$  of order 3 have the following characteristics:

- (1) One condition in 12 variables  $(t, x, u, v, u_x, v_x, \dots, u_{4x}, v_{4x})$ , two of which occur only explicitly  $(u_{4x}, v_{4x})$ , with 55 terms linear in functions  $P^t$  of eight variables  $(t, x, u, v, u_x, v_x, u_{xx}, v_{xx})$  and  $P^x$  of 10 variables  $(t, x, u, v, \dots, u_{xxx}, v_{xxx})$  and their first-order derivatives. The difference in the dependencies of  $P^t, P^x$  at the beginning of CONLAW1 is a consequence of using (38), (39) for substituting a first-order  $t$ -derivative of  $u$  by a second-order  $x$ -derivative.

<sup>†</sup>With the improved (September 1998) version of CRACK, times are reduced. For example, the 1977 seconds are now 76 seconds and the CONLAW1 second-order case can now be solved in 880 seconds.

**Table 1.** Run times  $t_1$  to formulate and  $t_2$  to solve determining conditions of conservation laws of the MVDNLS equations.

CONLAW	Order of $P^t$ for CONLAW1, order of $Q$ for CONLAW2/3									
	0		1		2		3		4	
	$t_1$	$t_2$	$t_1$	$t_2$	$t_1$	$t_2$	$t_1$	$t_2$	$t_1$	$t_2$
1	0.15	2.9	0.15	1977						
2	1.7	2.0	2.7	16	4.5	194	8.5	722	17	2784
3	0.17	4.5	0.18	11.7	0.3	28.5	0.6	377	1.9	low memory

- (2) One condition in 22 variables  $(t, x, u, v, \dots, u^{(3)}, v^{(3)})$ , six of which occur only explicitly (second-order derivatives of  $u_t, v_t$ ), with 37 terms linear in functions  $P^t, P^x$  of 14 variables  $(t, x, u, v, \dots, u^{(2)}, v^{(2)})$  and their first-order derivatives and, furthermore, functions  $Q^1, Q^2$  of 10 variables  $(t, x, u, v, \dots, u_{xxx}, v_{xxx})$ .
- (4) Two coupled conditions in 14 variables  $(t, x, u, v, u_x, v_x, \dots, u_{5x}, v_{5x})$ , four of which occur only explicitly  $(u_{4x}, v_{4x}, u_{5x}, v_{5x})$ , with 131 and 132 terms linear in functions  $Q^1, Q^2$  of 10 variables  $(t, x, u, v, \dots, u_{xxx}, v_{xxx})$  and their first- and second-order derivatives.

The following are general features of equations (1)–(4).

Equation (1) is of first order and therefore only highest-order  $u$ -derivatives which are not substituted due to  $\Delta = 0$  are not variables of the  $P^i$  and can be used for direct separation. Equation (1) therefore is only weakly over-determined with the application of integrability conditions playing an important role. A general problem with computing a differential Gröbner basis is that the complexity of these calculations depends heavily on the total ordering of derivatives of chosen functions  $P, Q$  for which there is currently no complete theory available. Choices made by the program can be particularly good or bad for the problem at hand.

In contrast, equations (4) are of higher order with more jet-variables that occur only explicitly and that can be used for direct separation. Although these equations are of a higher order they are highly over-determined and simpler to solve in general. An efficient way of doing direct separations and handling large equations is of importance for this approach.

Finally, in equations (2) the  $P^i$  depend initially on all jet-variables (apart from the highest-order  $u$ -derivatives), and also those substituted through  $\Delta = 0$  on which the  $Q^\mu$  do not depend. On the other hand, the  $Q^\mu$  do depend on the highest-order  $u$ -derivatives initially. The efficiency in solving (2) therefore depends on the efficiency of a module for indirect separation, i.e. on a module for handling equations which have no functions depending on all variables but which also have no variables occurring only explicitly so that no direct separation with respect to any variable is possible. Such a module is described in Wolf and Brand (1992).

All techniques are used to solve the over-determined system of all three approaches; only some are used more often in one approach than in the others.

There is another issue. If the order of derivatives with respect to different variables differs, such as, for example, in the Harry–Dym equation  $u_t - u^3 u_{xxx} = 0$ , then it matters whether this equation is used to perform the substitutions  $u_t = u^3 u_{xxx}$  or  $u_{xxx} = u_t / u^3$ . Substituting  $u_t$  gives a lower increase in complexity when successively higher-order ansätze for  $P$  or  $Q$  are made. On the other hand, one has to go to higher orders of  $P$  and  $Q$  to cover the same equivalence classes of conservation laws compared to

substituting  $u_{xxx}$ . As equations (4) already involve higher-order  $u$ -derivatives, a further increase could explode the size of (4) even more.

Another relation between (2) and (4) is that one could look at (4) as resulting from a differential Gröbner basis calculation done with (2), with the aim of eliminating the  $P^i$  first. It is, of course, more efficient to exploit knowledge of the structure of (2) and to apply the Euler operator to write down (4) directly rather than to do the differential Gröbner basis calculation step by step with (2). On the other hand, CRACK includes a number of modules to take advantage of special situations (e.g. to integrate exact PDEs or to recognize and solve PDEs that are ODEs for some partial derivatives and to solve them using ODESOLVE MacCallum (1988)). For a concrete problem it is very likely that there exists a quicker way to solve (2) than to first eliminate all  $P^i$ . The question of which of the CONLAW programs is best depends on the effectiveness of different submodules of the program CRACK which is used to solve conditions (1)–(4). With the current version of CRACK (January 1998), programs CONLAW1/3 are better for simpler conservation law problems while CONLAW2 is better for larger problems.

## 7. Syntax of CONLAW

EXAMPLE. The input to find conservation laws with  $Q$  of order 0–4 for the MVDNLS equations (38), (39) is:

```
depend u,x,t;
depend v,x,t;
conlaw2({{df(u,t) = - df( u*(u**2+v**2) + b*u - df(v,x) ,x),
          df(v,t) = - df( v*(u**2+v**2) + df(u,x)          ,x) },
        {u,v}, {t,x}
        },
        {0, 4, t, {}, {}});
```

In REDUCE, lists are enclosed in  $\{ \}$ . The input of CONLAW $i$  ( $i = 1, 2, 3$ ) consists of two lists, the first of which encodes the PDE problem. It contains a list of equations with the derivative to be substituted on the left-hand side, a list of functions and a list of independent variables. The second parameter to CONLAW $i$  is a list that specifies the conservation laws to be computed. Its first two elements are the minimum and maximum order of  $P^1$  in the case of CONLAW1 and the order of  $Q^\mu$  in the case of CONLAW2/3. The third element is `t` or `nil` and specifies whether the conservation law may depend explicitly on the  $x^i$  or not. The fourth element is a list of functions to be determined in an ansatz made for  $P^i$  or  $Q^\mu$  and the last element is a list of inequalities to be satisfied.

More details about investigating an ansatz is given in a manual file that comes with the three CONLAW $i$ .RED files.

## 8. Summary

Supplied with subroutines to fix gauge freedom in differential expressions, the programs CONLAW1/2/3 proved to be an efficient tool for the computation of the conservation laws of differential equations. Compared with other programs, (see, for instance, Göktaş and Hereman (1996)) the programs CONLAW $i$  show the following new features:

- By solving systems of over-determined differential equations, it is possible to find conservation laws with non-polynomial or even non-rational  $P, Q$ .



- It is possible to find conservation laws with an explicit dependence of  $P, Q$  on the independent variables.
- There is no limit to the number of DEs nor the number of independent variables to be investigated for conservation laws, other than the complexity of computations.
- It is possible to determine values of parameters in the DE such that conservation laws exist (examples in Wolf (1998)).
- For each of the programs CONLAWi, an ansatz for  $P^i$  and/or  $Q^\mu$  can be input to specify the conservation laws to be calculated.

Compared with the program of Göktaş and Hereman (1996), CONLAW is able to find more general conservation laws and to make a definitive statement on whether local conservation laws exist and whether the order is too high to complete the computations.

The strength of the program described in Göktaş and Hereman (1996) is that sometimes a higher order can still be handled by concentrating on polynomial conservation laws because the program only has to solve algebraic systems for coefficients of a polynomial ansatz. They were also able to extend applicability to differential-difference systems (Göktaş *et al.*, 1997; Göktaş and Hereman, forthcoming).

The comparison of the three approaches (1)–(4) showed that each of them has advantages in special circumstances. It also serves as a comparison between using a general purpose program to find the quickest way of solving over-determined PDE systems directly (CONLAW1/3) and an approach to derive integrability conditions by applying extra information about the structure of the PDE system (CONLAW2).

The programs including a manual and a test file are available via ftp from `ftp://ftp.maths.qmw.ac.uk/pub/tw`. The package will be submitted to the REDUCE network library.

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